Rendezvous Problems

by

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Rendezvous Problems

These talks will be roughly divided into the following topics:

1. Definitions
2. Ascent trajectories
3. Parking orbits
4. Mission analysis
5. Rendezvous in planetary travel

Definitions

The definitions of the major symbols are:

- a: semi-major axis
- E: eccentric anomaly
- V: true anomaly
- p: parameter of ellipse - semilatus rectum
- T: period
- i: inclination angle
- ω: argument of perigee
- Ω: argument of ascending node
- e: eccentricity

In terms of the quantities we will use the well-known relations:

1. $r_p = a(1 - e)$; pericenter distance
2. $r_a = a(1 + e)$; apocenter distance
3. $p = a(1 - e^2)$; semilatus rectum
4. $T = 2\pi a^{3/2} \mu^{-1/2}$,

   $\mu = 1.407639 \times 10^6 \text{ ft}^3/\text{sec}$ for the Earth's gravitational constant

5. $n = \overline{\omega} = \frac{2\pi}{T} = \mu^{1/2} a^{-3/2}$; mean motion

6. $M = n(t - \tau)$; mean anomaly
   τ = time at epoch.

If the Earth's potential function is represented by

7. $U = \frac{\mu}{r} \left[1 - \sum_{n=2}^{\infty} J_n \left(\frac{R}{r}\right)^n P_n (\sin L) \right]$

where

P_n = Legendre polynomial of order n

L = Latitude angle,

then the first order secular perturbations in the orbital elements of an Earth satellite in the absence of air drag are:

8. $\dot{\Omega}_s = - \frac{3}{2a} \sqrt{\frac{\mu}{a}}\, J_2 \left(\frac{R}{p}\right)^2 \cos i$ rad/sec ; R = equatorial radius

9. $\dot{\omega}_s = \frac{3}{4a} \sqrt{\frac{\mu}{a}}\, J_2 \left(\frac{R}{p}\right)^2 (- 1 + 5 \cos^2 i)$ rad/sec

10. $\dot{M}_s = \frac{3}{4a} \sqrt{\frac{\mu}{a}}\, J_2 \left(\frac{R}{p}\right)^2 \sqrt{1 - e^2}\, (- 1 + 3 \cos^2 i)$ rad/sec

11. $\dot{\Omega}_s = - 3 \pi J_2 \left(\frac{R}{p}\right)^2 \cos i$ rad/rev.

12. $\dot{\omega}_s = 3 \pi J_2 \left(\frac{R}{p}\right)^2 (- 1/2 + 5/2 \cos^2 i)$ rad/rev

13. $\dot{M}_s = 3 \pi J_2 \left(\frac{R}{p}\right)^2 \sqrt{1 - e^2}\, (- 1/4 + 3/4 \cos^2 i)$ rad/rev

where $J_2 = 1082.28 \times 10^{16}$.

Example:

For an orbit with $i = 30^{\circ}$ and an altitude of 300 statute miles, one finds that

$$\dot{\Omega} = -0.442\ ^{\circ}/\text{rev.} \approx -6.8\ ^{\circ}/\text{day}$$

$$\dot{\omega} = 0.705\ ^{\circ}/\text{rev.} \approx 10.8\ ^{\circ}/\text{day}$$

Rendezvous Phases

Rendezvous can be divided into the three phases

i. ascent of injection into transfer orbit
ii. terminal phase
iii. docking - contact between ferry and target vehicles.

There are a wide variety of possible types of ascent maneuvers and a few remarks will be made concerning the characteristics of some of the basic types of ascent maneuvers.

a. In - plane ascent:

An in - plane ascent requires that the target vehicle travel in a compatible orbit; that is, an orbit in which the target passes over the launch site at least once per day. This is a severe requirement and its practical realization will probably require means for adjusting the orbital period of the target vehicle.

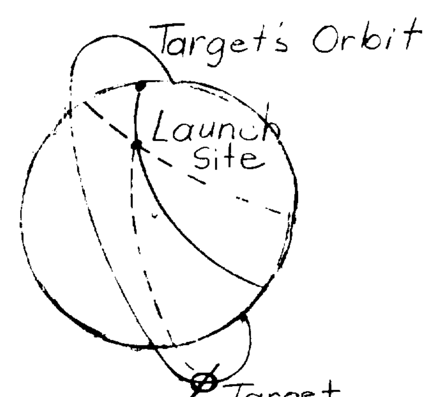


Figure 1
Transfer Orbit

b. Adjacency transfer:

The ferry is inserted into an orbit close to that of the target, but not necessarily in the same orbital plane. The ferry transfer orbit is selected so that its orbit is coaltitude and has the same velocity as the target at the time at which the two orbits intersect. At the time of orbit intersection, the ferry is given a velocity impulse such that its orbit plane is made coincident with that of the target.

c. Two - impulse transfer:

The first impulse inserts the ferry into a transfer orbit such that the apogee of the transfer occurs at the orbit of the ferry and the timing is such that the ferry and target are simultaneously at the apogee of the transfer orbit. When the two orbits touch, a second velocity impulse is given to the ferry to bring it up to orbital speed and, if necessary, change its orbital plane to coincide with that of the target.

d. General ascent:

The ferry is injected into a general transfer orbit which is required to intersect the target on either the outgoing leg or the incoming leg. The timing problem for these ascents is very critical and typical launch windows are only of about 3 minutes in duration.

e. Parking orbits:

An intermediate parking orbit greatly simplifies the timing problems for an ascent transfer trajector. The ferry is first launched into a circular orbit at a lower altitude than that of the target. Because the ferry will have a shorter period of revolution, it will gain on the target with respect to their geocentric angles. At the proper time, the ferry is given a velocity impulse into a transfer orbit which will bring it into position for the final rendezvous maneuver.

Velocity Penalty for Maneuvers

a. Equal velocities, in - plane maneuvers:

Suppose that the interceptor (ferry) and the target vehicles have the same velocity magnitude but different directions; Figure 2.

For small α ,

14. $\Delta V = \alpha V.$

For a typical velocity of 25,000 ft/sec, the velocity increment required per degree separation of the paths would be of the order

15. $\Delta V = \frac{\pi}{180} \times 25 \times 10^3 = 436$ ft/sec.

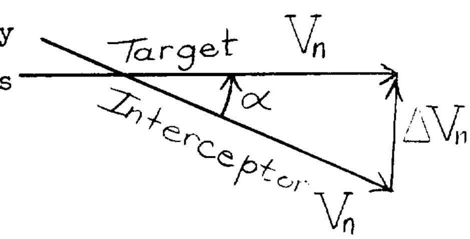


Figure 2

This is a costly maneuver as measured in units of required velocity impulse.

b. Two - impulse maneuver:

From Figure 2,

16. $V_2^2 = V_1^2 + V_0^2 - 2V_1V_0 \cos \alpha$.

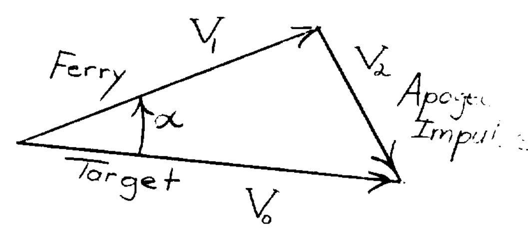


Figure 3

The velocity penalty for the plane change is

17. $\Delta V = V_1 + V_2 - V_0$,

for small α, such that $\sin\alpha \approx \alpha$, 16. and 17. yield

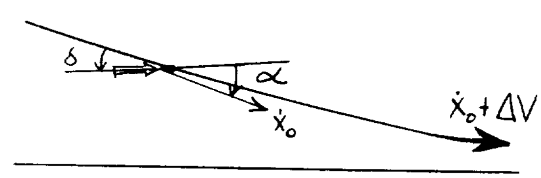
18. $$\Delta V = \frac{V_1 V_0}{2(V_0 - V_1)} \alpha^2.$$

Example:

Typical numbers at apogee are $V_1 = 10 \times 10^3$ ft/sec, $V_2 = 15 \times 10^3$, $V_0 = 25 \times 10^3$ ft/sec. If $\alpha = 5.7^\circ$, then $\Delta V \approx 83$ ft/sec. Thus the two-impulse maneuver is less costly than the previous case. The economy partly comes from the fact that the velocity impulse can correct the interceptor's speed at the same time that its orbital plane is shifted.

c. Dog - leg maneuvers:

Dog - leg maneuvers during thrusting may also be used during ascent or transfer trajectories to effect an orbital plane shift. Thrust is made in the transverse direction by tilting the rocket thrust by an angle δ from the vehicle's flight path. Let



ΔV = required increment of velocity.

Figure 4

It can be demonstrated that if $\dot{y} << \dot{x}_0$, then for δ held constant,

19. $$\frac{\dot{y}}{\dot{x}_0} = \alpha = \frac{\Delta V}{\dot{x}_0}\delta$$

Example:

If $\dot{x}_0 = 10 \times 10^3$ ft/sec, $\Delta V = 15 \times 10^3$ ft/sec, one finds that $\alpha = 1.5\,\delta$.

Thus the dog - leg maneuver can change the angle of the trajectory plane on the same order as the rocket motor gimbel angle used, and with minor penalty on the forward acceleration.

General Direct Ascent

a. General Direct Ascent

The rendezvous window is defined as the interval of time on the launch pad during which a rendezvous ascent can be made without an "excess" fuel penalty.

It has been established that Hohmann - type transfers produce minimum energy transfer. <u>Soft</u> - <u>rendezvous</u> is the situation in which the speed and orbit direction are the same for both the interceptor and target vehicles. A Hohmann - type transfer can be used if the target is at A_L at interceptor launch (ahead of insertion point). The intercept takes place at A_R.

The general cases occur for the target at either B_L (leading) or C_L (lagging) with the intercept accomplished at the intersection points B_R or C_R, respectively.

One can investigate the maximum spread in angle between initial points B_L and C_L which determine the allowable launch window with a restriction on the available $\triangle V$ capability of the interceptor.

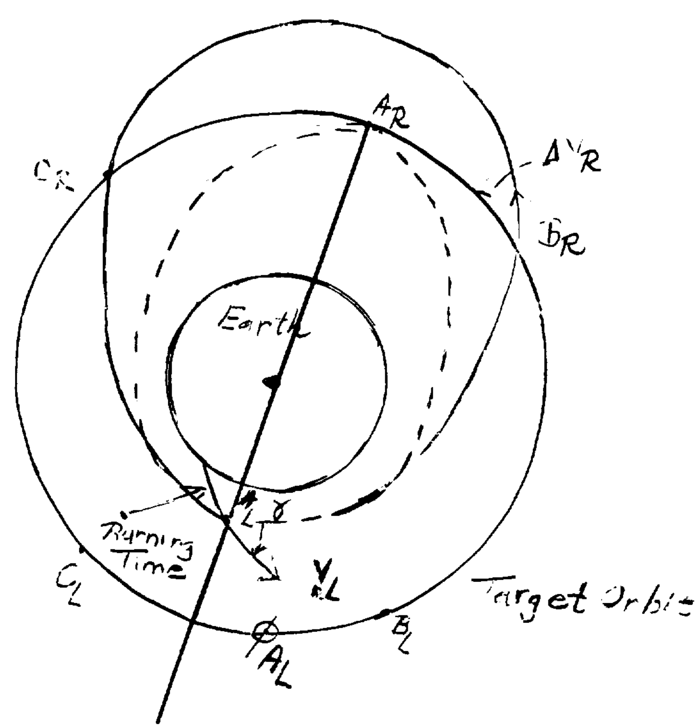


Figure 5
Transfer Orbits

Suppose that the total vehicle thrust capability is

$$20. \quad V_L + \Delta V_R = 27{,}000 \text{ ft/sec.}$$

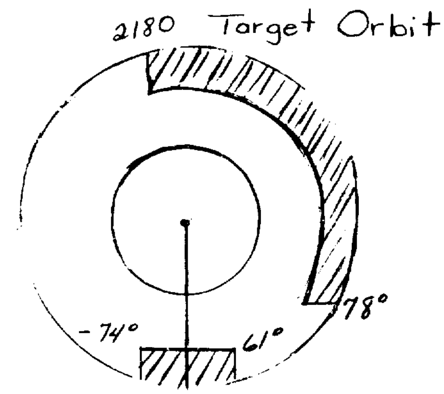


Figure 6

Launch Windows

One can show that the launch window shown in Figure 6, is - 7.4 to 6.1° or roughly 13°, which corresponds to about 3 minutes for typical orbits. If the thrust capability is increased to 3×10^4 ft/sec, the launch window increases to about 15 minutes. It is thus seen that the launch window is very sensitive to the total vehicle capability.

b. Indirect ascent schemes

Parking orbits can be employed to extend the launch windows from the order of minutes to hours.

Suppose, as shown in Figure 7, that the inclination of the target's orbital plane is only slightly larger than the latitude i_L of the launch site. Further, suppose that the interceptor is launched in a close orbit. That is, only a small angle change is required for rendevous. Let Δi denote the required difference in the inclination of the orbital plane. It can be demonstrated that

$$21. \quad \cos \theta = \frac{\sin i_L \cos i_0 - \sin \Delta i}{\cos i_L \sin i_0} \quad .$$

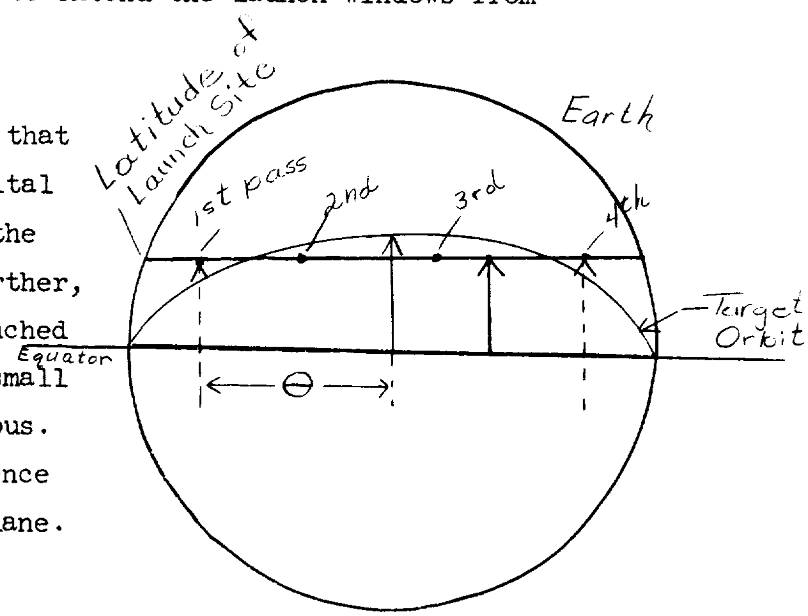


Figure 7

Example:

If $i_L = 28^{\circ}$ (Cape Canaveral)

i_0	Δ_i	θ
30°	2°	32.6°
30.4	2.4	36.0
31.0	3.0	39.5

Next consider two types of transfer orbits:

Case a: Transfer apogee at target height (Gemini Program Maneuver)

A chasing orbit is obtained by launching a transfer such that the apogee is tangent to the target's orbit. Thus the ferry or interceptor gains on the target during each revolution until a constellation is attained for which a single small impluse is sufficient to effect the rendezvous.

Let: θ = angular difference

ν = number of revolutions required to overcome θ deficiency

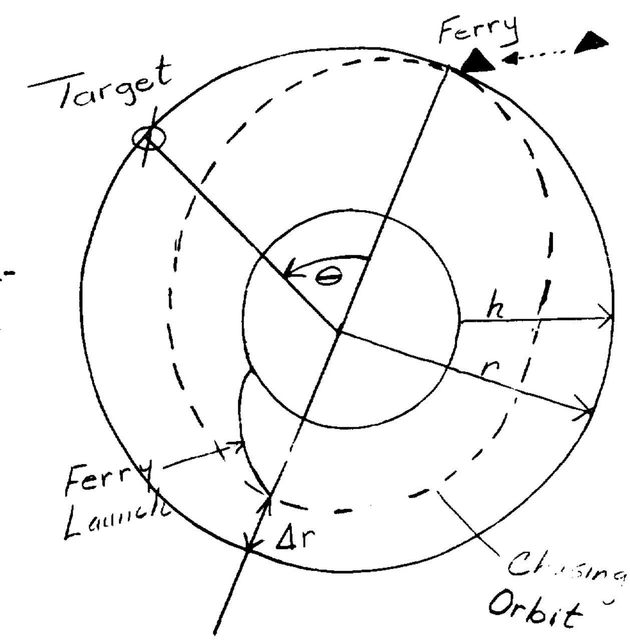


Figure 8

Hohmann - type Transfer

One can verify that

21. $$\frac{\Delta r}{r} = \frac{4}{3}\,\frac{\theta}{360\,\nu} \;,$$

22. $$\frac{\Delta V}{V_0} = \frac{1}{3}\;\frac{\theta}{360\,\nu} \;,$$

where V_0 is the orbital speed.

Example:

If $\theta = 20^\circ$

$\nu = 1$

$r = 4260$,

then $\Delta r = 315$. This cannot be accomplished in one revloution because Δr is greater than target altitude, here considered to be 300 s.m. Therefore, let $\nu = 2$, and then $\Delta r = 158$ miles and $\Delta V = 213$ ft/sec.

Case b: Parking orbit

For an intermediary parking orbit, 21. is modified to

$$23.\quad \frac{\Delta r}{r} = \frac{2}{3}\,\frac{\theta}{360(\nu - 3/4)}\,.$$

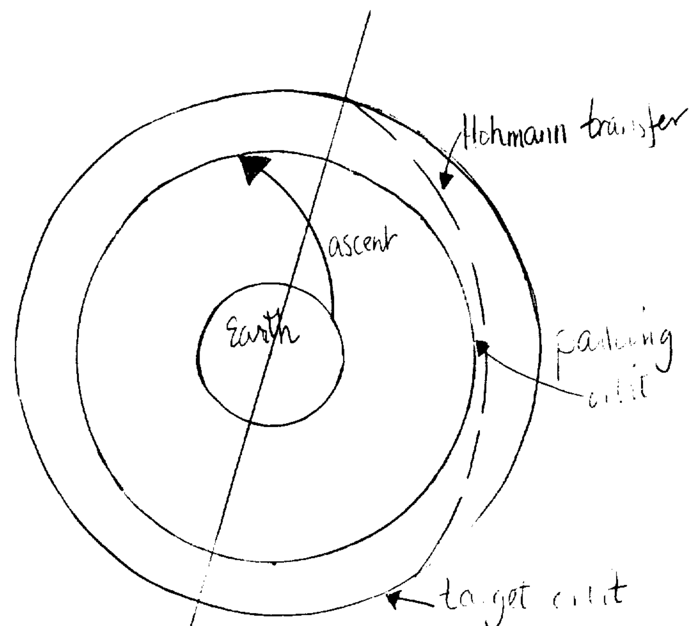


Figure 9

Thus the basic technique in the use of chasing or parking orbits is to launch the ferry any time it is ready during the time interval the launch site is close to the orbital plane of the target, Figure 7. From this figure and the table relating Δi and θ, this may be in the interval of several successive orbital passes. Any geocentric angular deficiency that the ferry may have is made up by use of the chasing or parking orbit. It is seen that the holding back for subsequent addition of a rendezvous velocity increment ΔV_R allows this type rendezvous to be made at substantially the same characteristic velocity increment as would be involved in a direct ascent rendezvous. These indirect schemes provide for launch windows up to 3-5 hours, instead of minutes.

Terminal Phase

Terminal phase starts when ferry is about 50 miles from the target.

Two types of terminal maneuvers are usually considered.

(i) Proportional navigation: maintain line-of-sight fixed in inertial space; or, maintain zero angular rate.

(ii) Orbital mechanics: compute coast orbits of target and ferry to determine if they intersect. If no intersection, compute required change in ferry orbit to produce orbit intersection.

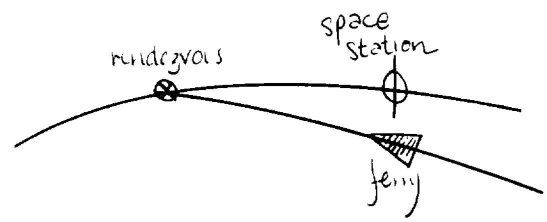


Figure 10

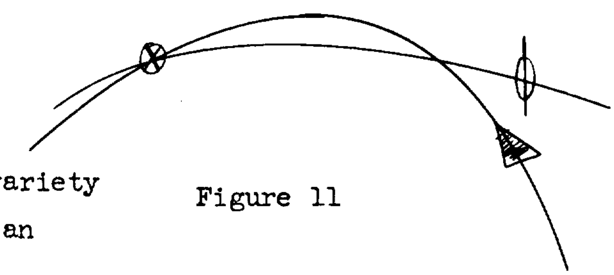


Figure 11

(a) Terminal Guidance

The terminal guidance equations for a variety of assumed models are given in Table 1. As an example, consider the equations in rotating rectangular coordinates for a model having a spherical earth, circular target orbit, "zero-order" gravity. The equations of motion are

$$24.\quad \ddot{x} - 2\omega\dot{y} = \frac{T_x}{m}$$
$$\ddot{y} + 2\omega\dot{x} - 3\omega^2 y = \frac{T_y}{m}$$
$$\ddot{z} + \omega^2 z = \frac{T_z}{m}$$

Assume no thrust, $T_x = T_y = T_z = 0$, then the solutions to 24. are

$$25.\quad x = \left(x_0 + \frac{2\dot{y}_0}{\omega}\right) + (-3\dot{x}_0 + 6\omega y_0)t - 2\left(3y_0 - 2\frac{\dot{x}_0}{\omega}\right)\sin\omega t - 2\frac{\dot{y}_0}{\omega}\cos\omega t,$$

26. $y = (4y_0 - 2\frac{\dot{x}_0}{\omega}) + (-3y_0 + 2\frac{\dot{x}_0}{\omega})\cos\omega t + \frac{\dot{y}_0}{\omega}\sin\omega t,$

27. $z = a_1 \sin\omega t + b_1 \cos\omega t,$

where x_0, $\dot{x}_0$, y_0, $\dot{y}_0$ are the initial conditions.

The general relative motion seen be the ferry in these coordinates is shown in Figure 12. The ellipse is centered at the target and has the following parameters:

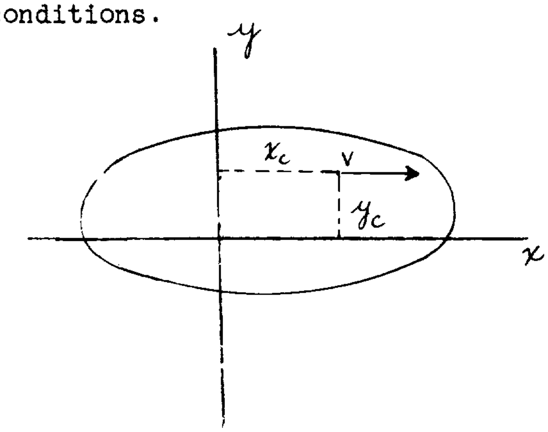


Figure 12

28. $v = -3\dot{x}_0 + 6\omega y_0$

$x_c = x_0 + 2\frac{\dot{y}_0}{\omega}$

$y_c = 4y_0 - 2\frac{\dot{x}_0}{\omega}$

$a = 2b$

$b = \left[(\frac{\dot{y}_0}{\omega})^2 + (3y_0 - 2\frac{\dot{x}_0}{\omega})^2 \right]^{\frac{1}{2}}.$

Suppose that $v = y_c = 0$; this implies that

29. $\dot{x}_0 = 2\omega y_0,$

which is the condition for which the orbital period of the transfer orbit is equal to that of the target.

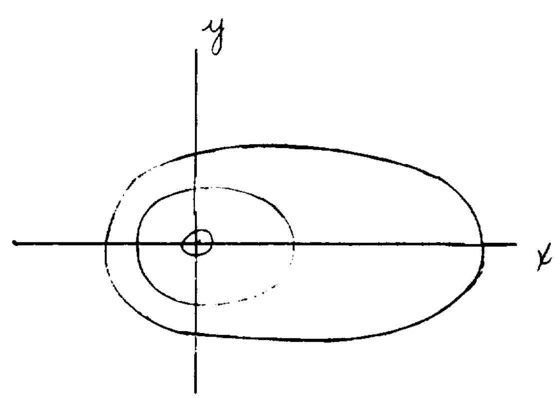


Figure 13

If

30. $\dot{x}_0 = 2\omega y_0$

$\dot{y}_0 = -\frac{\omega x_0}{2}$,

we have the situation illustrated in Figure 13, in which the ellipse is centered about the target.

If the target is itself in elliptical motion, it can be shown that the same form of terminal equations apply to the relative motion.

(b) Two Impulse Terminal Phase - Orbital Mechanics Scheme

Let $\omega/2\pi$ be the period of the target and t_r denote the time interval required to effect a rendezvous. Figures 15 and 16 illustrate the effect of the parameters required for a rendezvous.

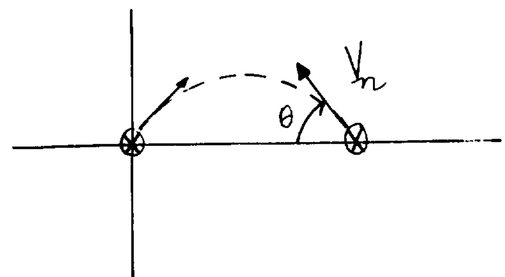


Figure 14

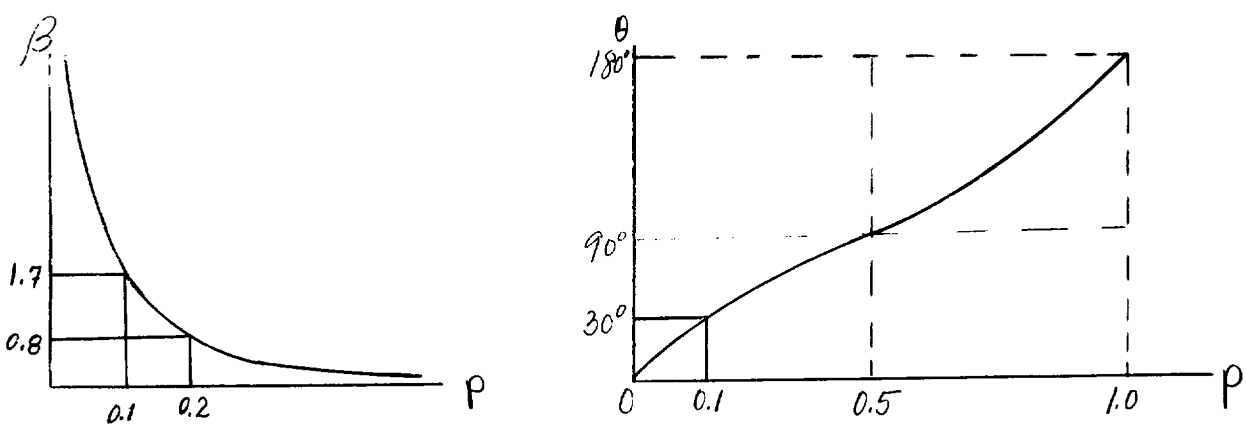


Figure 15

31. $p = \dfrac{\omega t_r}{2\pi}$, period ratio of the orbits,

ω = angular rate of orbit

$v = \beta \omega x_0$.

Example:

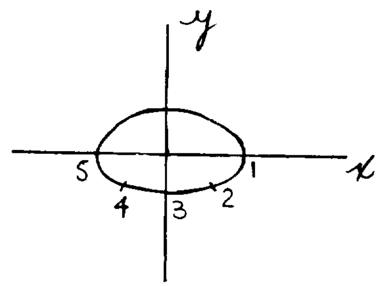
If the altitude h $\sim$200 miles, $\omega \approx 0.00114$.

For $\dot{x}_0$ = 5000 ft.

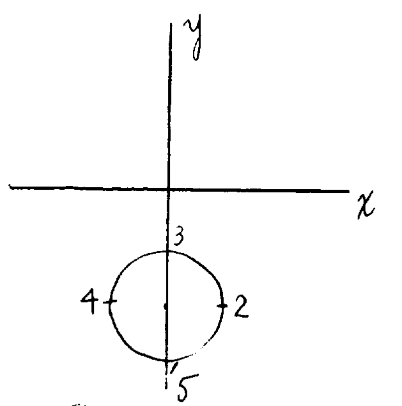
$v = 5.7 \beta$

If $\beta \approx 2$, $v \approx 10$ ft/sec to complete a rendezvous in a time t_r = 10 minutes.

(i) rotating axis system



(ii) inertial fixed system



Ferry Behind Space Station

(c) Proportional Navigation

Let R be the line-of-sight distance between the station and the vehicle. For proportional navigation, an intercept occurs if

32. $\dot{R}^2 = 2aR$

a = acceleration.

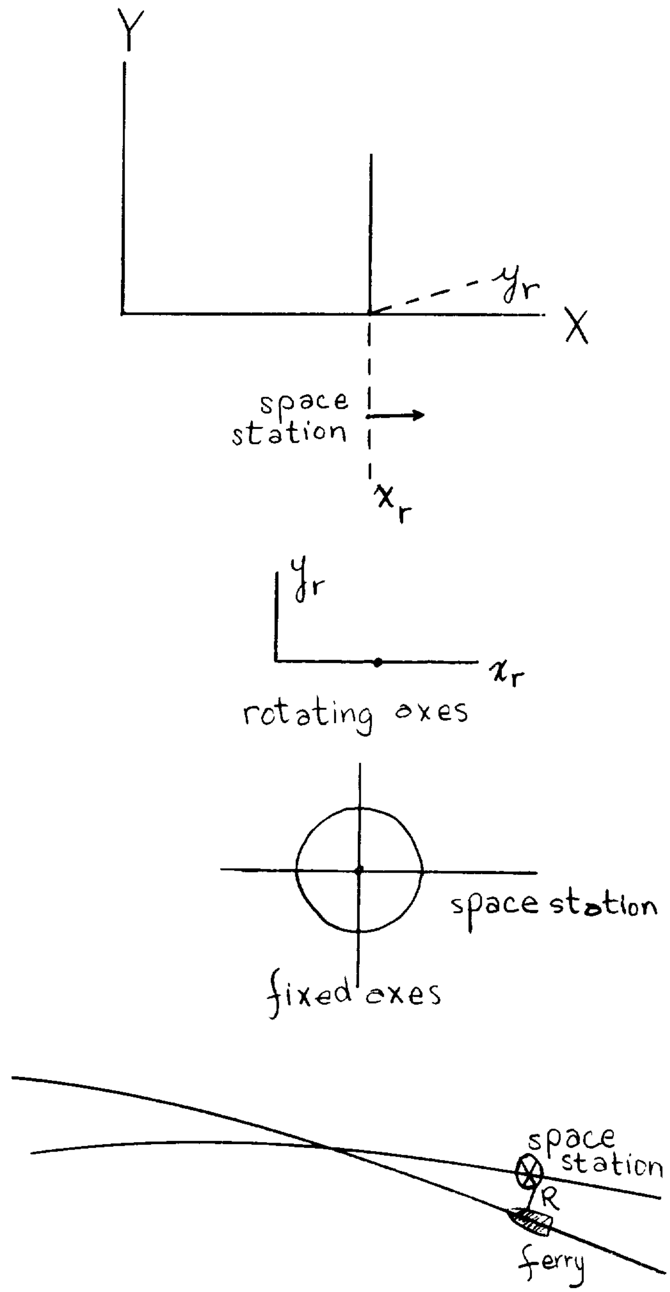


Figure 16

One maneuver for "braking" into a rendezvous is shown in Figure 17. A thrust is applied at the "on" line and removed at the "off" line. The vehicle then coasts until the "on" line is again met. The rendezvous is then made by "braking" in this stepwise fashion.

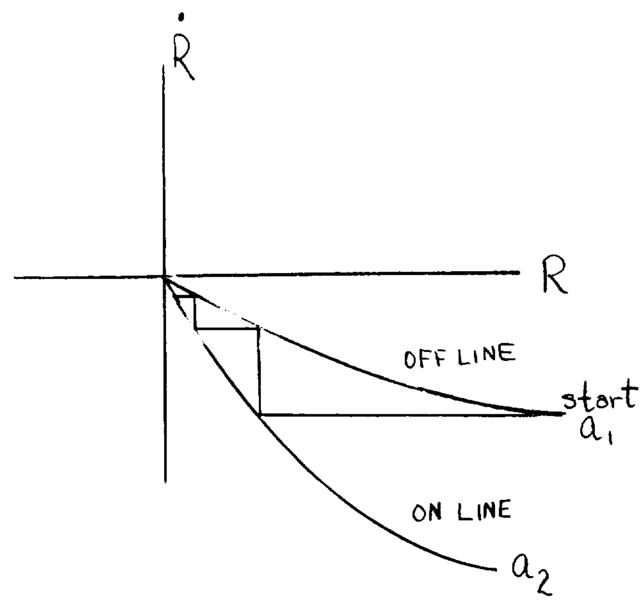


Figure 17

Mission Analysis

Mission analysis is used for booster design, or specifying the rocket thrust capabilities. As an example of mission design, consider a comparison of two types of lunar mission profiles:

(i) direct ascent

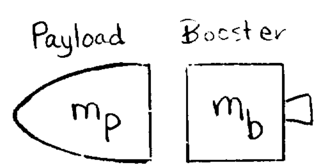
(ii) direct ascent with rendezvous in parking orbit about moon

The basic rocket equation can be written as

$$32a. \quad \frac{m_0}{m} = e^{\frac{\Delta v}{u}} = K$$

$$u = Ig$$

$$I = \text{specific impulse}$$



Consider the vehicle configuration of Figure 18. Apply 32a. to obtain

Figure 18

$$33.\quad \frac{m_p + m_b}{m_p + \epsilon\, m_b} = K,$$

where $\epsilon\, m_b$ = burn out weight of booster.

Solve for

$$34.\quad m_b = \frac{K - 1}{1 - \epsilon K} m_p = r m_p .$$

The required total weight is then

$$35.\quad m_T = m_p + m_b = \frac{(1 - \epsilon)K}{1 - \epsilon K} m_p = R m_p .$$

(a) Direct Ascent to Moon and Return

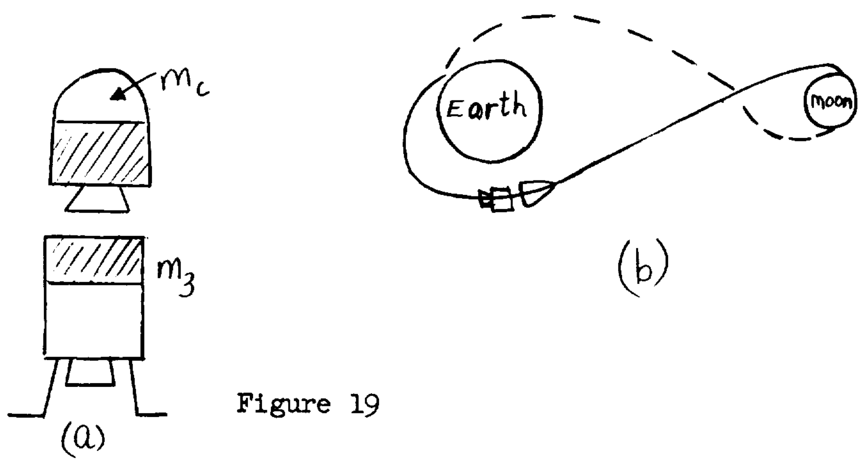


Figure 19

The sequence is

(i) Land on moon by means of m_3; ΔV_3 is the required velocity increment.

(ii) From the moon's surface, launch m_c to Earth return, ΔV_4 is the velocity increment.

Using the preceding equations, the mass that must be used to escape the Earth is

36. $M_e = R_3\,(m_T + m_3)$; $m_T = R_4 m_c = R_3\,(R_4 m_c + m_3)$

Typical velocity increments are:

$\Delta V_3 = 10{,}640$ ft/sec

$\Delta V_4 = 10{,}330$ ft/sec.

(b) Lunar Rendezvous

The mission sequence is

(i) Decelerate the vehicle into a moon orbit; ΔV_1

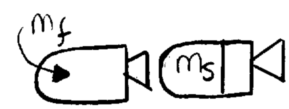


Figure 20

(ii) Descend to the moon with m_f, ascend to rendezvous; ΔV_d, ΔV_a.

(iii) Return to Earth; ΔV_2

One finds

37. $m = R_a\, m_f$,

38. $m_L = R_d\,(m + m_s) = R_d\,(R_a\, m_f + m_s)$.

It can be verified that the mass, m_e, that escapes from the Earth is

39. $m_e = R_{12}\,(\frac{m_L}{R_2} + m_c)$.

Typical velocity increments are:

$$\Delta V_a = 6800 \text{ ft/sec}$$

$$\Delta V_d = 6800$$

$$\Delta V_2 = 3530$$

$$\Delta V_{12} = \Delta V_1 + \Delta V_2 = 7370.$$

It is interesting to compare the two types of lunar profile missions. For a direct ascent with typical values

40. $M_e = 10\ m_c + 2.745\ m_s$

and

41. $m_e = 2.435\ m_c + 8.96\ m_f + 3.81\ m_s$

42. $m_L = 5.52\ m_f + 2.35\ m_s$.

If $m_c = 13{,}000$ lb, $m_f = 3500$, $m_s = 0$; one finds that

$$M_e = 130{,}000 \text{ lbs}$$

$$m_e = 64{,}000$$

$$m_L = 19{,}000.$$

These figures indicate the economy of a lunar rendezvous mission as compared to a direct ascent.

Rendezvous in Interplanetary Transfer

Rendezvous problems for interplanetary flights are exactly similar to those already discussed except in the near vicinity of the departing and destination planets. Figure 21 illustrates the hyperbolic escape orbit in the vicinity of the Earth.

The escape velocity is computed from

43. $$V^2 = \mu\left(\frac{2}{r} + \frac{1}{a}\right)$$

or, in equivalent form

44. $$V_E^2 = 2V_0^2 + V_\infty^2 \quad ,$$

where

V_0 = circular velocity at height r

V_∞ = hyperbolic excess velocity.

45. $$\tan\phi = \frac{V_E V_\infty}{V_0^2} \quad .$$

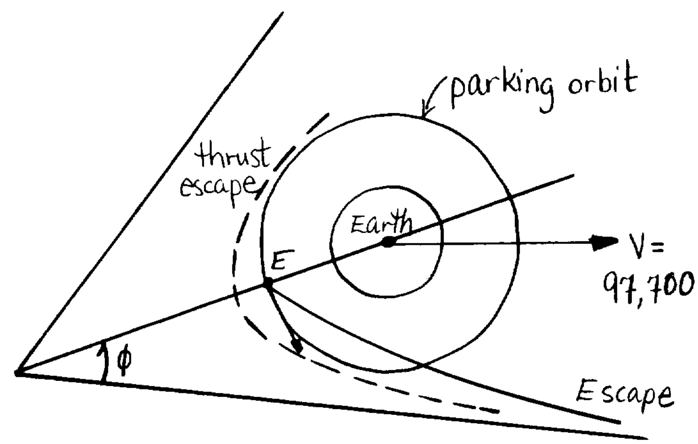


Figure 21

The thrust required for escape is computed from the Lagrangian

46. $$L = T - V = \tfrac{1}{2}\, m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{\mu}{r} \; ,$$

and

47. $$\delta\omega = F_r\,\delta r + F_\theta r\,\delta\theta .$$

The equations of motion for a thrusting escape are

48. $$\ddot{r} = r\dot{\theta}^2 + \frac{\mu}{mr^2} = \frac{u\dot{m}}{m}\frac{\dot{r}}{v}$$

49. $$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{\mu\dot{m}}{m}\frac{r\dot{\theta}}{V}$$

$$V = (\dot{r}^2 + r^2\dot{\theta}^2)^{\frac{1}{2}} \quad .$$

Instead of a Hohmann transfer, a faster orbit can be used as shown in Figure 22. The following table indicates the characteristics of a minimum energy Hohmann transfer to Mars compared to a possible fast orbit.

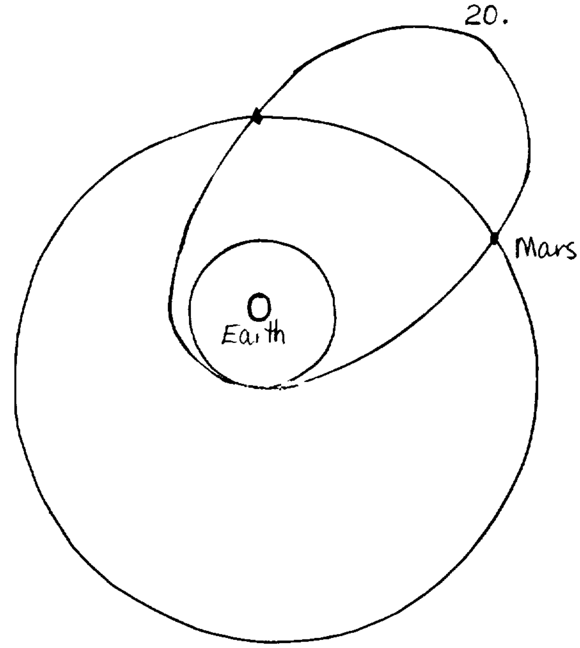


Figure 22

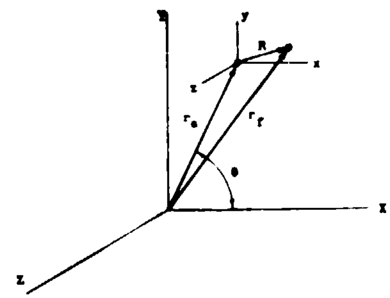
	Mars Stay Time	Travel Time	Total Time	$\sum \Delta V$
Hohmann	460 days	520 days	980 days	36,600 ft/sec
Fast Orbit	30	290	320	76,000

Reference:

J.C. Houbolt, "Problems and Potentialities of Space Rendezvous," Astronautica Acta, Volume VII, Fasc. 5-6, 1961.

Table 1. *Terminal Guidance Equations*

Inertially fixed axes



	Vector form	Rectangular coordinates
Exact	$\dfrac{\partial^2 \bar{r}_f}{\partial t^2} - \nabla \dfrac{\mu}{r_f} = \dfrac{\bar{T}}{m}$	Similar to form immediately below except in the expansion of $\nabla(\mu/r_f)$
Spherical earth	$\dfrac{d^2 \bar{r}_f}{dt^2} + \dfrac{GM}{r_f^3}\bar{r}_f = \dfrac{\bar{T}}{m}$ $\bar{r}_f = \bar{r}_s + \bar{R}$ $\bar{r}_s = (X, Y, Z)$ $= (r_s \cos\theta, r_s \sin\theta, 0)$ $\bar{R} = (x, y, z)$	$\ddot{x} + (\ddot{r}_s - r_s\dot{\theta}^2)\cos\theta - (2\dot{r}_s\dot{\theta} + r_s\ddot{\theta})\sin\theta + \dfrac{GM}{r_f^3}(x + r_s\cos\theta) = \dfrac{T_x}{m}$ $\ddot{y} + (\ddot{r}_s - r_s\dot{\theta}^2)\sin\theta + (2\dot{r}_s\dot{\theta} + r_s\ddot{\theta})\cos\theta + \dfrac{GM}{r_f^3}(y + r_s\sin\theta) = \dfrac{T_y}{m}$ $\ddot{z} + \dfrac{GM}{r_f^3}z = \dfrac{T_z}{m}$

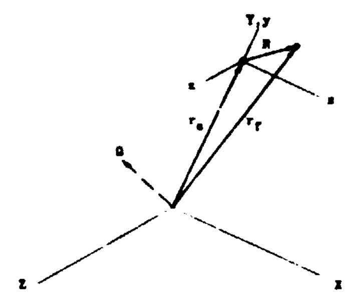
	Rotating set of axes	
		
Comments	**Vector form**	**Rectangular coordinates**
$\bar{r}_f = \bar{r}_s + \bar{R}$ μ/r_f is the gravity potential due to earth, moon, planets etc.	$\frac{\partial^2 \bar{r}_f}{\partial t^2} + 2\bar{\Omega} \times \frac{\partial \bar{r}_f}{\partial t} + \dot{\bar{\Omega}} \times \bar{r}_f + \bar{\Omega} \times \bar{\Omega} \times \bar{r}_f - \nabla \frac{\mu}{r_f} = \frac{\bar{T}}{m}$	Similar to form immediately below except in the expansion of $\nabla(\mu/r_f)$
θ is the angular velocity of station about center of earth, r_s is radial position of station	$\frac{d^2 \bar{r}_f}{dt^2} + 2\bar{\Omega} \times \frac{d\bar{r}_f}{dt} + \dot{\bar{\Omega}} \times \bar{r}_f + \bar{\Omega} \times \bar{\Omega} \times \bar{r}_f + \frac{GM}{r_f^3}\bar{r}_f = \frac{\bar{T}}{m}$ $\bar{r}_f = (x, y + r_s, z) = \bar{r}_s + \bar{R}$ $\bar{r}_s = (0, r_s, 0)$ $\bar{R} = (x, y, z)$ $\bar{\Omega} = (0, 0, \dot{\theta})$	$\ddot{x} - (y + r_s)\ddot{\theta} - 2(\dot{y} + \dot{r}_s)\dot{\theta} - x\left(\dot{\theta}^2 - \frac{GM}{r_f^3}\right) = \frac{T_x}{m}$ $\ddot{y} + r\ddot{\theta} + 2\dot{x}\dot{\theta} + \ddot{r}_s - (y + r_s)\left(\dot{\theta}^2 - \frac{GM}{r_f^3}\right) = \frac{T_y}{m}$ $\ddot{z} + \frac{GM}{r_f^3} z = \frac{T_z}{m}$ EGGLESTON [4, 5]

Table I (continued)

	Vector form	Rectangular coordinates
Spherical earth, Station in a circular orbit	$\frac{d^2\bar{R}}{dt^2} + \frac{d^2\bar{r}_s}{dt^2} + \frac{GM}{r_f^3}\bar{r}_f = \frac{\bar{T}}{m}$	$\ddot{x} - r_s\omega^2\cos\theta + \frac{GM}{r_f^3}(x + r_s\cos\theta) = \frac{T_x}{m}$ $\ddot{y} - r_s\omega^2\sin\theta + \frac{GM}{r_f^3}(y + r_s\sin\theta) = \frac{T_y}{m}$ $\ddot{z} + \frac{GM}{r_f^3}z = \frac{T_z}{m}$
Spherical earth, Circular orbit, 1st order gravity field	$\frac{d^2\bar{R}}{dt^2} + \frac{GM}{r_s^3}\left(\bar{R} - 3\frac{\bar{r}_s\cdot\bar{R}}{r_s^2}\bar{r}_s\right) = \frac{\bar{T}}{m}$ HORD [9], KURBJUN [10], BRISSENDEN [11]	$\ddot{x} + \omega^2(x - 3x\cos^2\theta - 3y\sin\theta\cos\theta) = \frac{T_x}{m}$ $\ddot{y} + \omega^2(y - 3x\sin\theta\cos\theta - 3y\sin^2\theta) = \frac{T_y}{m}$ $\ddot{z} + \omega^2 z = \frac{T_z}{m}$
Spherical earth, Circular orbit, "Zero-order" gravity		
No gravity	$\frac{d^2\bar{R}}{dt^2} = \frac{\bar{T}}{m}$ any orbit HORD [9]	Circular orbit $\ddot{x} = \frac{T_x}{m}$ $\ddot{y} = \frac{T_y}{m}$ $\ddot{z} = \frac{T_z}{m}$

Comments	Vector form	Rectangular Coordinates
$r_s =$ Constant $\dot{\theta} =$ Constant $= \omega$ $\bar{\Omega} =$ Constant $= (0, 0, \omega) = \bar{\omega}$ $\omega^2 = \frac{GM}{r_s^3} = \dot{\theta}^2$	$\frac{d^2\bar{R}}{dt^2} + 2\bar{\omega} \times \frac{d\bar{R}}{dt} + \bar{\omega} \times \bar{\omega} \times \bar{r}_f + \frac{GM}{r_f^3}\bar{r}_f = \frac{\bar{T}}{m}$ or $\frac{d^2\bar{R}}{dt^2} + 2\bar{\omega} \times \frac{d\bar{R}}{dt} + \omega^2 \bar{k} z + \left(\frac{GM}{r_f^3} - \omega^2\right)\bar{r}_f = \frac{\bar{T}}{m}$	$\ddot{x} - 2\omega\dot{y} - x\left(\omega^2 - \frac{GM}{r_f^3}\right) = \frac{T_x}{m}$ $\ddot{y} + 2\omega\dot{x} - (y + r_s)\left(\omega^2 - \frac{GM}{r_f^3}\right) = \frac{T_y}{m}$ $\ddot{z} + \frac{GM}{r_f^3} z = \frac{T_z}{m}$ EGGLESTON [4, 5]
$\frac{d^2\bar{r}_s}{dt^2} = -\omega^2\bar{r}_s;\ \frac{GM}{r_s^3} = \omega^2$ On the left, $\frac{GM}{r_f^3} = \frac{GM}{r_s^3}\left(1 - 3\frac{\bar{r}_s \cdot \bar{R}}{r_s^2} + \ldots\right)$ $\frac{GM}{r_f^3}\bar{r}_f \cong \omega^2\left(\bar{r}_s + \bar{R} - 3\frac{\bar{r}_s \cdot \bar{R}}{r_s^2}\bar{r}_s\right)$ On the right, $\frac{GM}{r_f^3} = \frac{GM}{r_s^3}\left(1 - 3\frac{y}{r_s} + \ldots\right)$ $\frac{GM}{r_f^3}\bar{r}_f \cong \omega^2(\bar{r}_s + \bar{R} - \bar{j}\,3y)$	$\frac{d^2\bar{R}}{dt^2} + 2\bar{\omega} \times \frac{d\bar{R}}{dt} + \bar{\omega} \times \bar{\omega} \times \bar{R} + \frac{GM}{r_s^3}(\bar{R} - \bar{j}\,3y) = \frac{\bar{T}}{m}$ or $\frac{d^2\bar{R}}{dt^2} + 2\bar{\omega} \times \frac{d\bar{R}}{dt} + \frac{GM}{r_s^3}(-\bar{j}\,3y + \bar{k}z) = \frac{\bar{T}}{m}$	$\left.\begin{aligned}\ddot{x} - 2\omega\dot{y} &= \frac{T_x}{m}\\ \ddot{y} + 2\omega\dot{x} - 3\omega^2 y &= \frac{T_y}{m}\\ \ddot{z} + \omega^2 z &= \frac{T_z}{m}\end{aligned}\right\}\ (T\,1)$ WHEELON [14], CLOHESSY and WILSHIRE [15], CARNEY [17], EGGLESTON [5, 18], SPRALIN [19]
$\frac{GM}{r_f^3}\bar{r}_f \cong \frac{GM}{r_s^3}(\bar{r}_s + \bar{R})$ $\frac{GM}{r_s^3} = \omega^2$	$\frac{d^2\bar{R}}{dt^2} + 2\bar{\omega} \times \frac{d\bar{R}}{dt} + \bar{\omega} \times \bar{\omega} \times \bar{R} + \frac{GM}{r_s^3}\bar{R} = \frac{\bar{T}}{m}$ or $\frac{d^2\bar{R}}{dt^2} + 2\bar{\omega} \times \frac{d\bar{R}}{dt} + \frac{GM}{r_s^3}\bar{k}z = \frac{\bar{T}}{m}$	$\ddot{x} - 2\omega\dot{y} = \frac{T_x}{m}$ $\ddot{y} + 2\omega\dot{x} = \frac{T_y}{m}$ $\ddot{z} + \omega^2 z = \frac{T_z}{m}$ HORNBY [20]
	$\frac{d^2\bar{r}_f}{dt^2} + 2\bar{\Omega} \times \frac{d\bar{r}_f}{dt} + \dot{\bar{\Omega}} \times \bar{r}_f + \bar{\Omega} \times \bar{\Omega} \times \bar{r}_f = \frac{\bar{T}}{m}$ for any orbit $\frac{d^2\bar{R}}{dt^2} + 2\bar{\omega} \times \frac{d\bar{R}}{dt} + \bar{\omega} \times \bar{\omega} \times \bar{r}_f = \frac{\bar{T}}{m}$ for circular orbit	Circular orbit $\ddot{x} - 2\omega\dot{y} - \omega^2 x = \frac{T_y}{m}$ $\ddot{y} + 2\omega x - \omega^2(y + r_s) = \frac{T_y}{m}$ $\ddot{z} = \frac{T_z}{m}$